RGPVNOTES.IN

Subject Name: **Strength of materials**

Subject Code: **ME-3002**

Semester: **3rd**

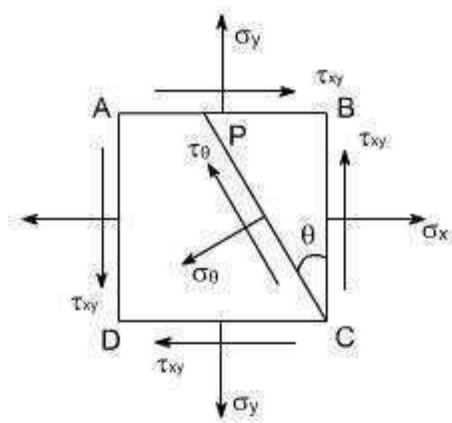
Strength of Material (ME-3002) Class Notes

UNIT II

Material subjected to combined direct and shear stresses:

Now consider a complex stress system shown below, acting on an element of material.

The stresses s_x and s_y may be compressive or tensile and may be the result of direct forces or as a result of bending. The shear stresses may be as shown or completely reversed and occur as a result of either shear force or torsion as shown in the figure below:



As per the double subscript notation the shear stress on the face BC should be notified as t_{yx} , however, we have already seen that for a pair of shear stresses there is a set of complementary shear stresses generated such that $t_{yx} = t_{xy}$

By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

(i) Material subjected to pure state of stress shear. In this case the various formulas deserved are as follows

$s_q = t_{yx}$ sin2 q

$t_q = -\ t_{yx}$ cos 2 q

(ii) Material subjected to two mutually perpendicular direct stresses. In this case the various formulas's derived are as follows.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

To get the required equations for the case under consideration, let us add the respective equations for the above two cases such that

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

These are the equilibrium equations for stresses at a point. They do not depend on material proportions and are equally valid for elastic and inelastic behaviour

This eqn. gives two values of 2q that differ by 180^0 .Hence the planes on which maximum and minimum normal stresses occur at 90^0 apart.

For σ_θ to be a maximum or minimum $\dfrac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$

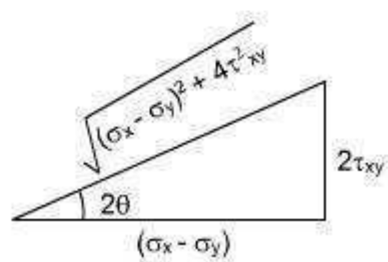
$$= 0$$

$$\text{i.e.} -(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$$

$$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$$

$$\text{Thus,} \qquad \boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$$

From the triangle it may be determined



$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

Substituting the values of cos 2q and sin 2q in equation (5) we get

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}.\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} + \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}.\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} + \frac{1}{2}\frac{4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

or

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}.\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}.\frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Hence we get the two values of σ_θ, which are designated σ_1 as σ_2 and respectively,therefore

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

The σ_1 and σ_2 are termed as the principle stresses of the system.

Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (6) we see that

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = 0$$

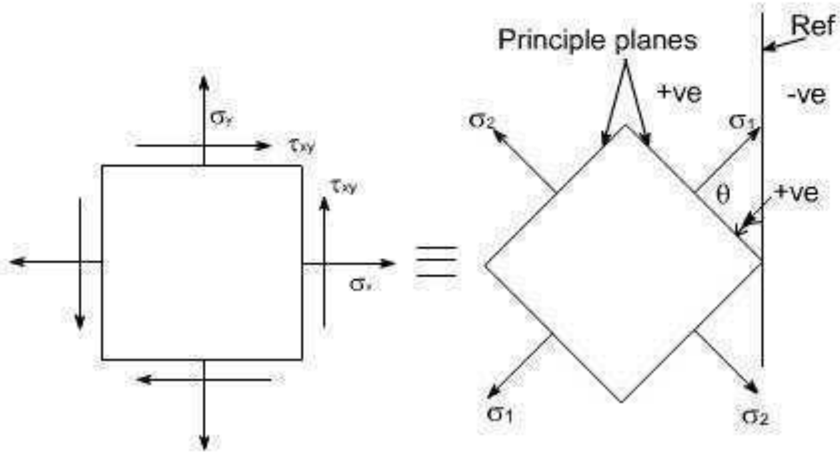
This shows that the values of shear stress are zero on the principal planes.

Hence the maximum and minimum values of normal stresses occur on planes of zero shearing stress. The maximum and minimum normal stresses are called the principal stresses, and the planes on which they act are called principal plane the solution of equation

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

will yield two values of 2q separated by 180^0 i.e. two values of q separated by 90^0 .Thus the two principal stresses occur on mutually perpendicular planes termed principal planes.

Therefore the two – dimensional complex stress system can now be reduced to the equivalent system of principal stresses.



Let us recall that for the case of a material subjected to direct stresses the value of maximum shear stresses

$$\tau_{max^m} = \frac{1}{2}(\sigma_x - \sigma_y) \text{ at } \theta = 45^0$$ Thus, for a 2-dimensional state of stress, subjected to principle stresses

$$\tau_{max^m} = \frac{1}{2}(\sigma_1 - \sigma_2)$$, on substituting the values if σ_1 and σ_2, we get

$$\tau_{max^m} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Alternatively this expression can also be obtained by differentiating the expression for τ_θ with respect to θ i.e.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta.2 + \tau_{xy}\sin 2\theta.2$$

$$= 0$$

$$\text{or } (\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_s = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Recalling that

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Thus,

$$\boxed{\tan 2\theta_p.\tan 2\theta_s = 1}$$

Therefore, it can be concluded that the equation (2) is a negative reciprocal of equation (1) hence the roots for the double angle of equation (2) are 90^0 away from the corresponding angle of equation (1).

This means that the angles that angles that locate the plane of maximum or minimum shearing stresses form angles of 45^0 with the planes of principal stresses.

Further, by making the triangle we get

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

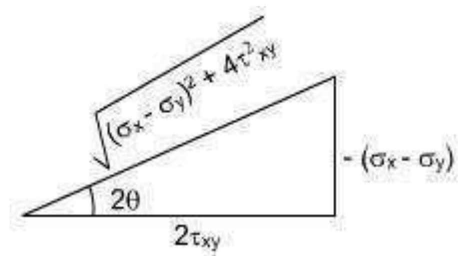
Therefore by substituting the values of $\cos 2\theta$ and $\sin 2\theta$ we have

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}.-\frac{(\sigma_x - \sigma_y).(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$= -\frac{1}{2}.\frac{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$



Because of root the difference in sign convention arises from the point of view of locating the planes on which shear stress act. From physical point of view these sign have no meaning.

The largest stress regard less of sign is always knows as maximum shear stress.

Principal plane inclination in terms of associated principal stress:

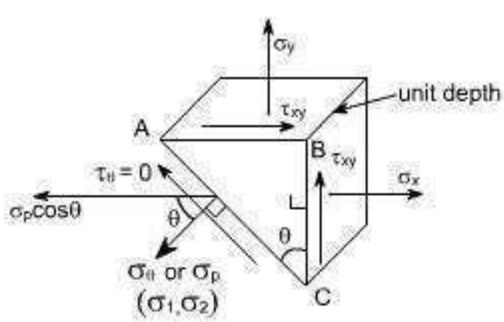
We know that the equation $\tan 2\theta_p = \dfrac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$

Yields two values of q i.e. the inclination of the two principal planes on which the principal stresses s_1 and s_2 act. It is uncertain, however, which stress acts on which plane unless equation.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

is used and observing which one of the two principal stresses is obtained.

Alternatively we can also find the answer to this problem in the following manner



Consider once again the equilibrium of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses s_p acts, and the shear stress is zero.

Resolving the forces horizontally we get:

s_x .BC . 1 + t_{xy} .AB . 1 = s_p . cosq . AC dividing the above equation through by BC we get

$$\sigma_x . + \tau_{xy}\frac{AB}{BC} = \sigma_p . \cos\theta . \frac{AC}{BC}$$

or

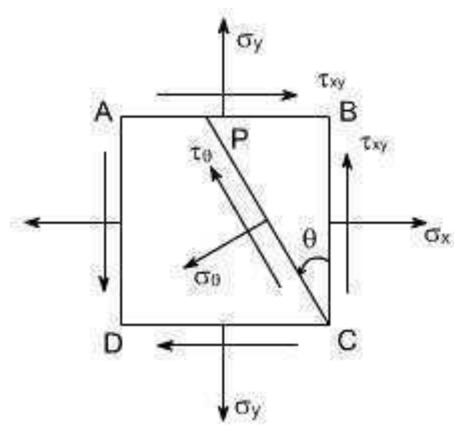
$$\sigma_x + \tau_{xy}\tan\theta = \sigma_p$$

Thus

$$\boxed{\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}}$$

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This graphical representation is very useful in depending the relationships between normal and shear stresses acting on any inclined plane at a point in a stresses body.

To draw a Mohr's stress circle consider a complex stress system as shown in the figure



The above system represents a complete stress system for any condition of applied load in two dimensions

The Mohr's stress circle is used to find out graphically the direct stress s and sheer stress t on any plane inclined at q to the plane on which s_x acts. The direction of q here is taken in anticlockwise direction from the BC.

STEPS:

In order to do achieve the desired objective we proceed in the following manner

(i) Label the Block ABCD.

(ii) Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)

(iii) Plot the stresses on two adjacent faces e.g. AB and BC, using the following sign convention.

Direct stresses - tensile positive; compressive, negative

Shear stresses – tending to turn block clockwise, positive

– tending to turn block counter clockwise, negative

[i.e shearing stresses are +ve when its movement about the center of the element is clockwise]

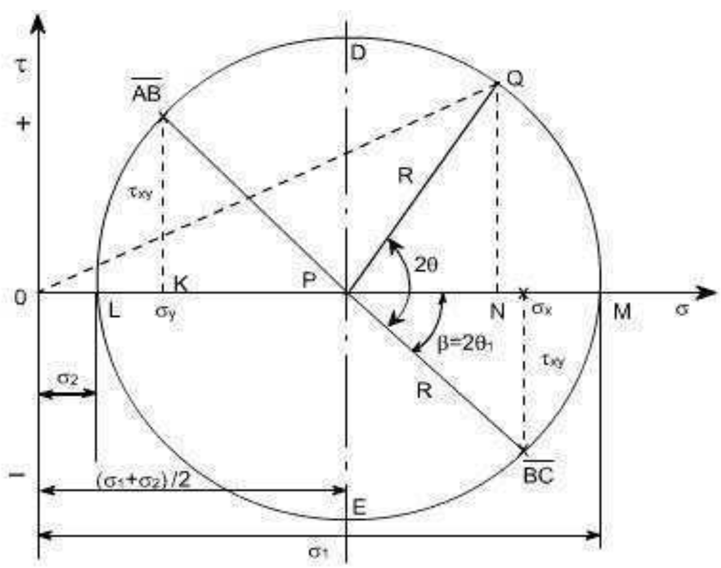
This gives two points on the graph which may than be labeled as $\overline{AB}$ and $\overline{BC}$ respectively to denote stresses on these planes.

(iv) Join $\overline{AB}$ and $\overline{BC}$.

(v) The point P where this line cuts the s axis is than the centre of Mohr's stress circle and the line joining $\overline{AB}$ and $\overline{BC}$ is diameter. Therefore the circle can now be drawn.

Now every point on the circle then represents a state of stress on some plane through C.

Proof:



Consider any point Q on the circumference of the circle, such that PQ makes an angle 2q with BC, and drop a perpendicular from Q to meet the s axis at N. Then OQ represents the resultant stress on the plane an angle q to BC. Here we have assumed that $s_x > s_y$

Now let us find out the coordinates of point Q. These are ON and QN.

From the figure drawn earlier

ON = OP + PN

OP = OK + KP

$OP = s_y + 1/2\ (s_x - s_y)$

$= s_y / 2 + s_y / 2 + s_x / 2 + s_y / 2$

$= (s_x + s_y) / 2$

PN = Rcos(2q - b)

Hence ON = OP + PN

= ($s_x + s_y$) / 2 + Rcos(2q - b)

= ($s_x + s_y$) / 2 + Rcos2q cosb + Rsin2qsinb

now make the substitutions for Rcosb and Rsinb.

$$R\cos\beta = \frac{(\sigma_x - \sigma_y)}{2};\ R\sin\beta = \tau_{xy}$$

Thus,

ON = 1/2 ($s_x + s_y$) + 1/2 ($s_x - s_y$)cos2q + t_{xy}sin2q

Similarly QM = Rsin(2q - b)

= Rsin2qcosb - Rcos2qsinb

Thus, substituting the values of R cosb and Rsinb, we get

QM = 1/2 ($s_x - s_y$)sin2q - t_{xy}cos2q

If we examine above equations see that this is the same equation which we have already derived analytically

Thus the co-ordinates of Q are the normal and shear stresses on the plane inclined at q to BC in the original stress system.

N.B: Since angle $\overline{BC}$ PQ is 2q on Mohr's circle and not q it becomes obvious that angles are doubled on Mohr's circle. This is the only difference, however, as they are measured in the same direction and from the same plane in both figures.

Further points to be noted are:

(1) The direct stress is maximum when Q is at M and at this point obviously the sheer stress is zero, hence by definition OM is the length representing the maximum principal stresses s_1 and $2q_1$ gives the angle of the plane q_1 from BC. Similar OL is the other principal stress and is represented by s_2

(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle.

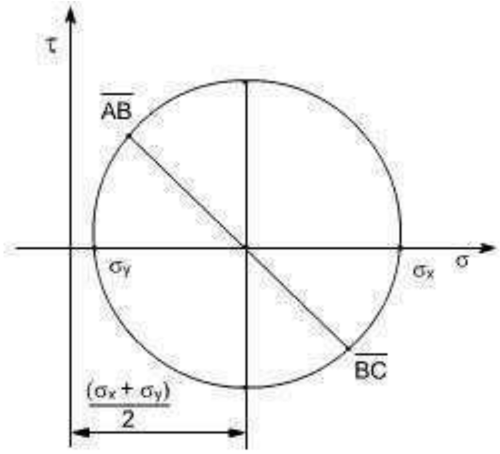
This follows that since shear stresses and complimentary sheer stresses have the same value; therefore the centre of the circle will always lie on the s axis midway between s_x and s_y . [since +t_{xy} & -t_{xy} are shear stress & complimentary shear stress so they are same in magnitude but different in sign.]

(3) From the above point the maximum sheer stress i.e. the Radius of the Mohr's stress circle would be

$$\frac{(\sigma_x - \sigma_y)}{2}$$

While the direct stress on the plane of maximum shear must be mid – may between s_x and s_y i.e

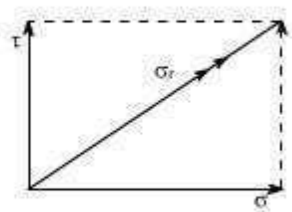
$$\frac{(\sigma_x + \sigma_y)}{2}$$



(4) As already defined the principal planes are the planes on which the shear components are zero.

Therefore are concluding that on principal plane the sheer stress is zero.

(5) Since the resultant of two stresses at 90^0 can be found from the parallogram of vectors as shown in the diagram. Thus, the resultant stress on the plane at q to BC is given by OQ on Mohr's Circle.



(6) The graphical method of solution for a complex stress problems using Mohr's circle is a very powerful technique, since all the information relating to any plane within the stressed element is contained in the single construction. It thus, provides a convenient and rapid means of solution, which is less prone to arithmetical errors and is highly recommended.

ILLUSRATIVE PROBLEMS:

Let us discuss few representative problems dealing with complex state of stress to be solved either analytically or graphically.

PROB 1: A circular bar 40 mm diameter carries an axial tensile load of 105 kN. What is the Value of shear stress on the planes on which the normal stress has a value of 50 MN/m^2 tensile?

Solution:

Tensile stress s_y= F / A = 105 x 10^3 / p x $(0.02)^2$

= 83.55 MN/m^2

Now the normal stress on an oblique plane is given by the relation

$s_q = s_y \sin^2 q$

$50 \times 10^6 = 83.55$ MN/m^2 $\times 10^6 \sin^2 q$

q = 50^{0}68'

The shear stress on the oblique plane is then given by

$t_q = 1/2\ s_y \sin 2q$

$= 1/2 \times 83.55 \times 10^6 \times \sin 101.36$

$= 40.96\ MN/m^2$

Therefore the required shear stress is 40.96 MN/m^2

PROB 2:

For a given loading conditions the state of stress in the wall of a cylinder is expressed as follows:

(a) 85 MN/m^2 tensile

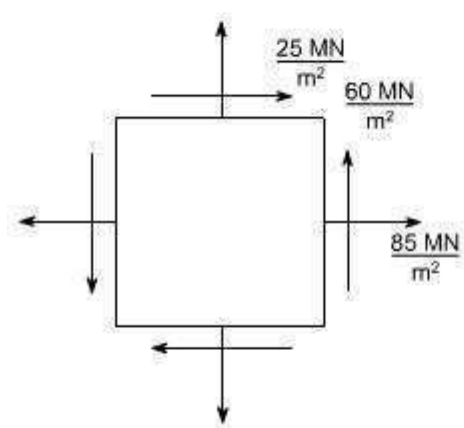
(b) 25 MN/m^2 tensile at right angles to (a)

(c) Shear stresses of 60 MN/m^2 on the planes on which the stresses (a) and (b) act; the sheer couple acting on planes carrying the 25 MN/m^2 stress is clockwise in effect.

Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged?

Solution:

The problem may be attempted both analytically as well as graphically. Let us first obtain the analytical solution



The principle stresses are given by the formula

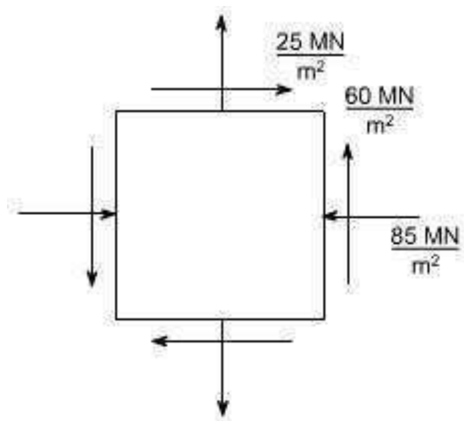
$$\begin{aligned}
&\sigma_1 \text{ and } \sigma_2 \\
&= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}} \\
&= \frac{1}{2}(85 + 25) \pm \frac{1}{2}\sqrt{(85 + 25)^2 + (4 \times 60^2)} \\
&= 55 \pm \frac{1}{2}.60\sqrt{5} = 55 \pm 67 \\
&\Rightarrow \sigma_1 = 122 \text{ MN/m}^2 \\
&\quad \sigma_2 = -12 \text{ MN/m}^2 (\text{compressive})
\end{aligned}$$

For finding out the planes on which the principle stresses act us the equation

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

The solution of this equation will yield two values q i.e. they q_1 and q_2 giving $q_1 = 31^0 71'$ & $q_2 = 121^0 71'$

(b) In this case only the loading (a) is changed i.e. its direction had been changed. While the other stresses remains unchanged hence now the block diagram becomes.



Again the principal stresses would be given by the equation.

$$\sigma_1 \,\&\, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$= \frac{1}{2}(-85 + 25) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= \frac{1}{2}(-60) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= -30 \pm \frac{1}{2}\sqrt{12100 + 14400}$$

$$= -30 \pm 81.4$$

$$\sigma_1 = 51.4 \text{ MN/m}^2;\ \sigma_2 = -111.4 \text{ MN/m}^2$$

Again for finding out the angles use the following equation.

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

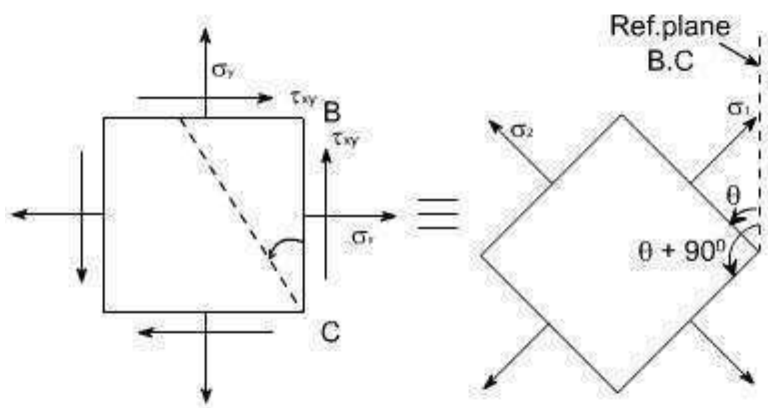
$$= \frac{2 \times 60}{-85 - 25} = +\frac{120}{-110}$$

$$= -\frac{12}{11}$$

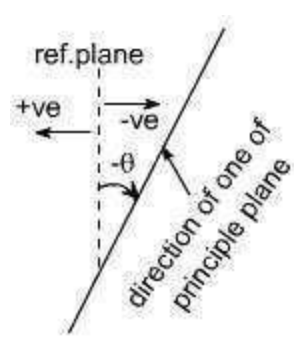
$$2\theta = \tan\left(-\frac{12}{11}\right)$$

$$\Rightarrow \theta = -23.74^0$$

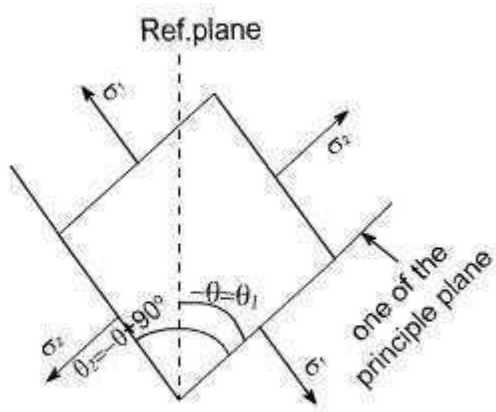
Thus, the two principle stresses acting on the two mutually perpendicular planes i.e. principle planes may be depicted on the element as shown below:



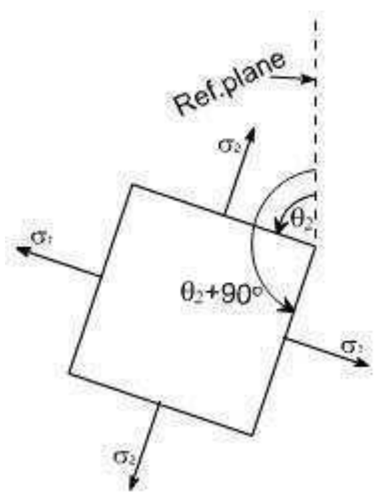
So this is the direction of one principle plane & the principle stresses acting on this would be s_1 when is acting normal to this plane, now the direction of other principal plane would be 90^0 + q because the principal planes are the two mutually perpendicular plane, hence rotate the another plane q + 90^0 in the same direction to get the another plane, now complete the material element if q is negative that means we are measuring the angles in the opposite direction to the reference plane BC .



Therefore the direction of other principal planes would be {-q + 90} since the angle -q is always less in magnitude then 90 hence the quantity (-q + 90) would be positive therefore the Inclination of other plane with reference plane would be positive therefore if just complete the Block. It would appear as



If we just want to measure the angles from the reference plane, than rotate this block through 180^0 so as to have the following appearance.

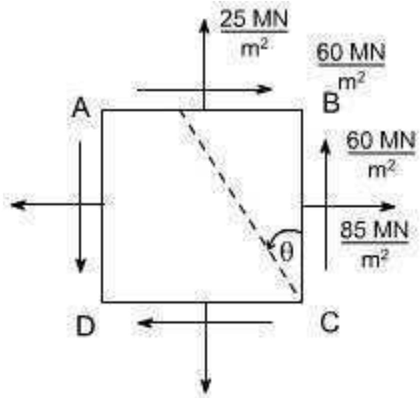


So whenever one of the angles comes negative to get the positive value,
first Add 90^0 to the value and again add 90^0 as in this case q = $-23^0 74'$
so $q_1 = -23^0 74' + 90^0 = 66^0 26'$ .Again adding 90^0 also gives the direction of other principle planes i.e $q_2 = 66^0 26' + 90^0 = 156^0 26'$

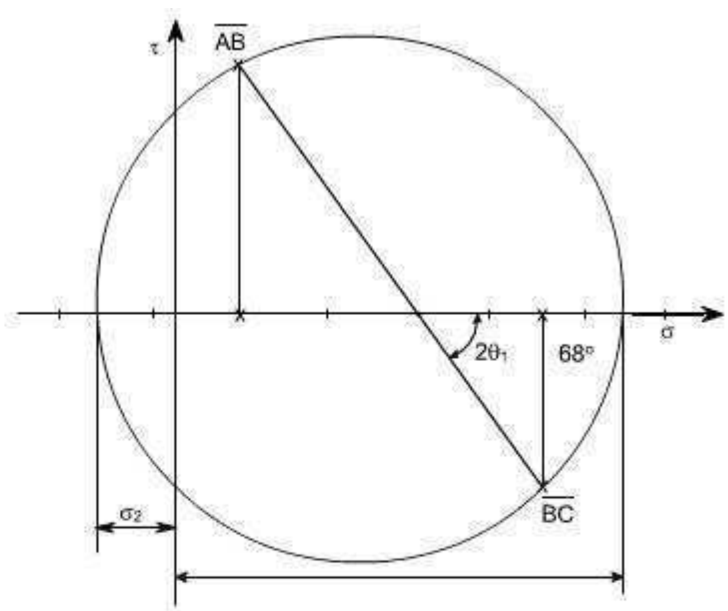
This is how we can show the angular position of these planes clearly.

GRAPHICAL SOLUTION:

Mohr's Circle solution: The same solution can be obtained using the graphical solution i.e. the Mohr's stress circle, for the first part, the block diagram becomes



Construct the graphical construction as per the steps given earlier.



Taking the measurements from the Mohr's stress circle, the various quantities computed are

s_1 = 120 MN/m^2 tensile

s_2 = 10 MN/m^2 compressive

q_1 = 34^0 counter clockwise from BC

q_2 = $34^0 + 90 = 124^0$ counter clockwise from BC

Part Second: The required configuration i.e. the block diagram for this case is shown along with the stress circle.

By taking the measurements, the various quantities computed are given as

s_1 = 56.5 MN/m^2 tensile

s_2 = 106 MN/m^2 compressive

q_1 = $66^0$15' counter clockwise from BC

$q_2 = 156^0 15'$ counter clockwise from BC

Salient points of Mohr's stress circle:

1. Complementary shear stresses (on planes 90^0 apart on the circle) are equal in magnitude

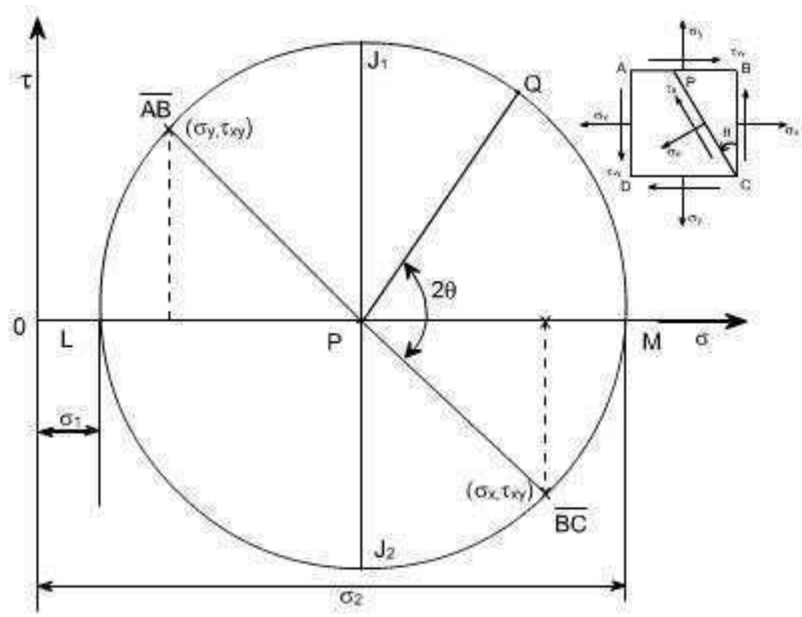
2. The principal planes are orthogonal: points L and M are 180^0 apart on the circle (90^0 apart in material)

3. There are no shear stresses on principal planes: point L and M lie on normal stress axis.

4. The planes of maximum shear are 45^0 from the principal points D and E are 90^0 , measured round the circle from points L and M.

5. The maximum shear stresses are equal in magnitude and given by points D and E

6. The normal stresses on the planes of maximum shear stress are equal i.e. points D and E both have normal stress co-ordinate which is equal to the two principal stresses.



As we know that the circle represents all possible states of normal and shear stress on any plane through a stresses point in a material. Further we have seen that the co-ordinates of the point 'Q' are seen to be the same as those derived from equilibrium of the element. i.e. the normal and shear stress components on any plane passing through the point can be found using Mohr's circle. Worthy of note:

1. The sides AB and BC of the element ABCD, which are 90^0 apart, are represented on the circle by $\overline{AB}$ P and $\overline{BC}$ P and they are 180^0 apart.

2. It has been shown that Mohr's circle represents all possible states at a point. Thus, it can be seen at a point. Thus, it, can be seen that two planes LP and PM, 180^0 apart on the diagram and therefore 90^0 apart in the material, on which shear stress t_q is zero. These planes are termed as principal planes and normal stresses acting on them are known as principal stresses.

Thus , s_1 = OL

s_2 = OM

3. The maximum shear stress in an element is given by the top and bottom points of the circle i.e by points J_1 and J_2 ,Thus the maximum shear stress would be equal to the radius of i.e. $t_{max}= 1/2(s_1- s_2)$,the corresponding normal stress is obviously the distance OP = $1/2 (s_x+ s_y)$, Further it can also be seen that the planes on which the shear stress is maximum are situated 90^0 from the principal planes (on circle), and 45^0 in the material.

4.The minimum normal stress is just as important as the maximum. The algebraic minimum stress could have a magnitude greater than that of the maximum principal stress if the state of stress were such that the centre of the circle is to the left of origin.

i.e. if $s_1 = 20$ MN/m^2 (say)

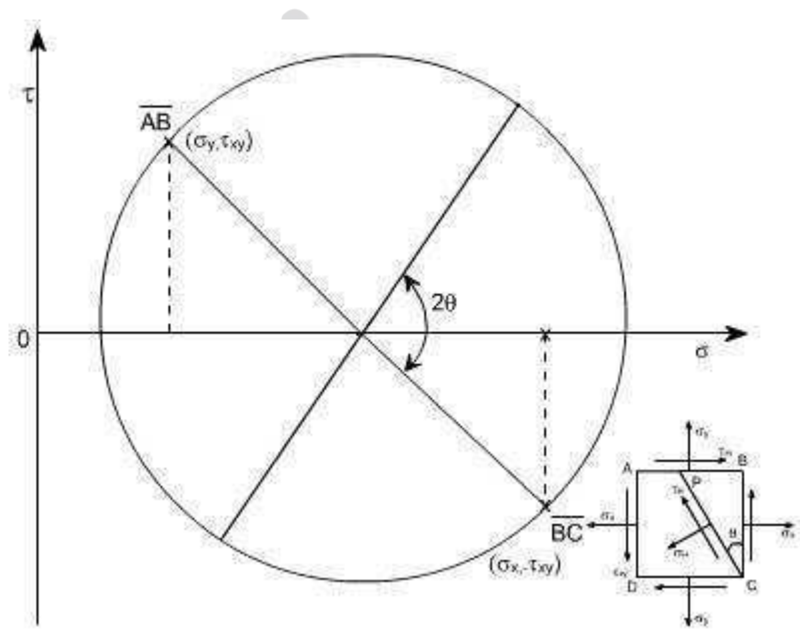
$s_2 = -80$ MN/m^2 (say)

Then $t_{max}^m = (s_1 - s_2 / 2) = 50$ MN/m^2

If should be noted that the principal stresses are considered a maximum or minimum mathematically e.g. a compressive or negative stress is less than a positive stress, irrespective or numerical value.

5. Since the stresses on perpendicular faces of any element are given by the co-ordinates of two diametrically opposite points on the circle, thus, the sum of the two normal stresses for any and all orientations of the element is constant, i.e. thus sum is an invariant for any particular state of stress.

Sum of the two normal stress components acting on mutually perpendicular planes at a point in a state of plane stress is not affected by the orientation of these planes.



This can be also understand from the circle Since AB and BC are diametrically opposite thus, whatever may be their orientation, they will always lie on the diameter or we can say that their sum won't change, it can also be seen from analytical relations

We know $\sigma_n = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy} \sin 2\theta$

on plane BC; q = 0

$s_{n1} = s_x$

on plane AB; q = 270^0

$s_{n2} = s_y$

Thus $s_{n1} + s_{n2} = s_x + s_y$

6. If $s_1 = s_2$, the Mohr's stress circle degenerates into a point and no shearing stresses are developed on xy plane.

7. If $s_x + s_y = 0$, then the center of Mohr's circle coincides with the origin of s - t co-ordinates.

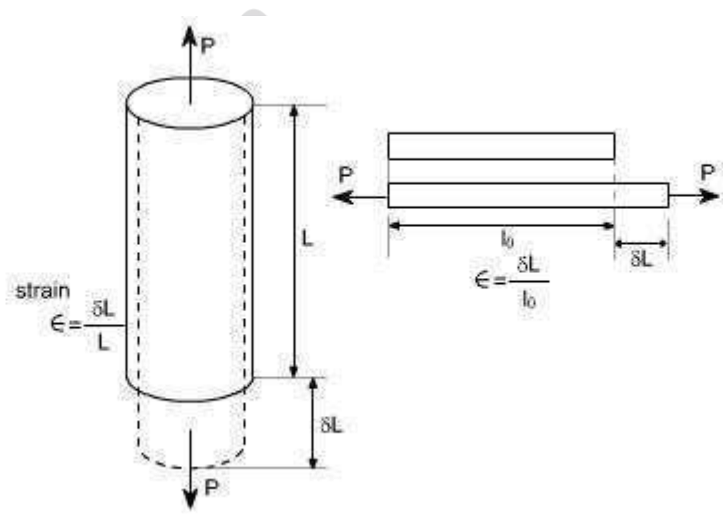
ANALYSIS OF STRAINS

CONCEPT OF STRAIN

Concept of strain : if a bar is subjected to a direct load, and hence a stress the bar will change in length. If the bar has an original length L and changes by an amount dL, the strain produce is defined as follows:

$$\text{strain}(\epsilon) = \frac{\text{change in length}}{\text{orginal length}} = \frac{\delta L}{L}$$

Strain is thus, a measure of the deformation of the material and is a no dimensional Quantity i.e. it has no units. It is simply a ratio of two quantities with the same unit.



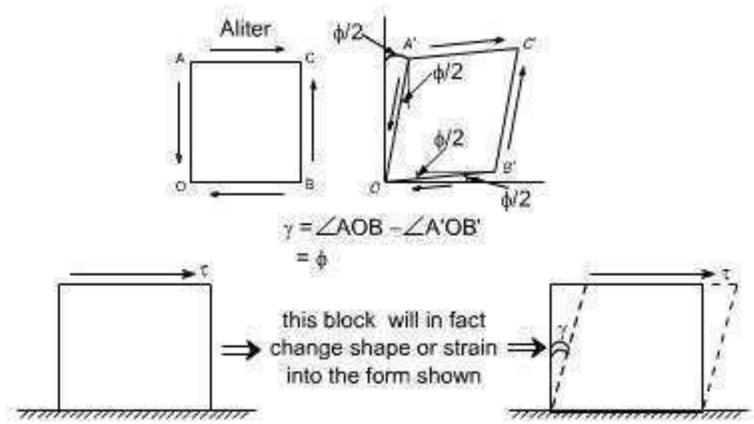
Since in practice, the extensions of materials under load are very very small, it is often convenient to measure the strain in the form of strain x 10^{-6} i.e. micro strain, when the symbol used becomes m Î.

Sign convention for strain:

Tensile strains are positive whereas compressive strains are negative. The strain defined earlier was known as linear strain or normal strain or the longitudinal strain now let us defines the shear strain.

Definition: An element which is subjected to a shear stress experiences a deformation as shown in the figure below. The tangent of the angle through which two adjacent sides rotate relative to their initial position is termed shear strain. In many cases the angle is very small and the angle itself is used, (in radians), instead of tangent, so that g = Ð AOB - Ð A'OB' = f

Shear strain: As we know that the shear stresses acts along the surface. The action of the stresses is to produce or being about the deformation in the body consider the distortion produced b shear sheer stress on an element or rectangular block



This shear strain or slide is f and can be defined as the change in right angle. Or The angle of deformation g is then termed as the shear strain. Shear strain is measured in radians & hence is non – dimensional i.e. it has no unit.So we have two types of strain i.e. normal stress & shear stresses.

Hook's Law :

A material is said to be elastic if it returns to its original, unloaded dimensions when load is removed.

Hook's law therefore states that

Stress (s) a strain(Î)

$$\boxed{\frac{\text{stress}}{\text{strain}} = \text{constant}}$$

Modulus of elasticity: Within the elastic limits of materials i.e. within the limits in which Hook's law applies, it has been shown that

Stress / strain = constant

This constant is given by the symbol E and is termed as the modulus of elasticity or Young's modulus of elasticity

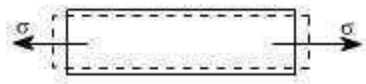
$$E = \frac{\text{strain}}{\text{stress}} = \frac{\sigma}{\epsilon}$$

$$= \frac{P/A}{\delta L/L}$$

Thus $\boxed{E = \frac{PL}{A\delta L}}$

The value of Young's modulus E is generally assumed to be the same in tension or compression and for most engineering material has high, numerical value of the order of 200 GPa.

Poisson's ratio: If a bar is subjected to a longitudinal stress there will be a strain in this direction equal to s / E . There will also be a strain in all directions at right angles to s . The final shape being shown by the dotted lines.

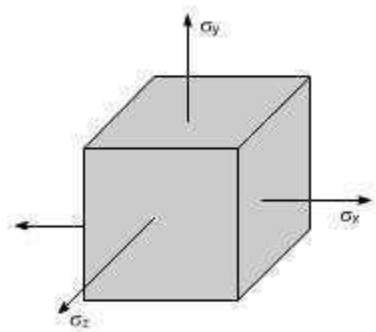


It has been observed that for elastic materials, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to longitudinal strain is known as the poison's ratio.

Poison's ratio (m) = - lateral strain / longitudinal strain

For most engineering materials the value of m his between 0.25 and 0.33.

Three – dimensional state of strain: Consider an element subjected to three mutually perpendicular tensile stresses s_x , s_y and s_z as shown in the figure below.



If s_y and s_z were not present the strain in the x direction from the basic definition of Young's modulus of Elasticity E would be equal to

$Î_x$= s_x/ E

The effects of s_y and s_z in x direction are given by the definition of Poisson's ratio ‘ m ' to be equal as -m s_y/ E and -m s_z/ E

The negative sign indicating that if s_y and s_z are positive i.e. tensile, these they tend to reduce the strain in x direction thus the total linear strain is x direction is given by

$$\epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}$$
$$\epsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_z}{E}$$
$$\epsilon_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E}$$

Principal strains in terms of stress:

In the absence of shear stresses on the faces of the elements let us say that s_x , s_y , s_z are in fact the principal stress. The resulting strain in the three directions would be the principal strains.

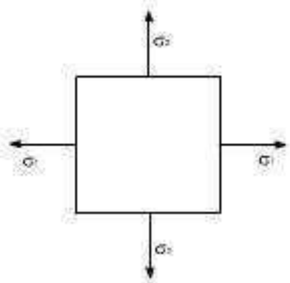
$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\sigma_2 - \mu\sigma_3\right]$$
$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\sigma_1 - \mu\sigma_3\right]$$
$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\sigma_1 - \mu\sigma_2\right]$$

For Two dimensional strain: system, the stress in the third direction becomes zero i.e $s_z = 0$ or $s_3 = 0$

Although we will have a strain in this direction owing to stresses s_1& s_2 .

Hence the set of equation as described earlier reduces to

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \mu\sigma_2]$$
$$\epsilon_2 = \frac{1}{E}[\sigma_2 - \mu\sigma_1]$$
$$\epsilon_3 = \frac{1}{E}[-\mu\sigma_1 - \mu\sigma_2]$$



Hence a strain can exist without a stress in that direction

$$\text{i.e if } \sigma_3 = 0;\ \epsilon_3 = \frac{1}{E}[-\mu\sigma_1 - \mu\sigma_2]$$

Also

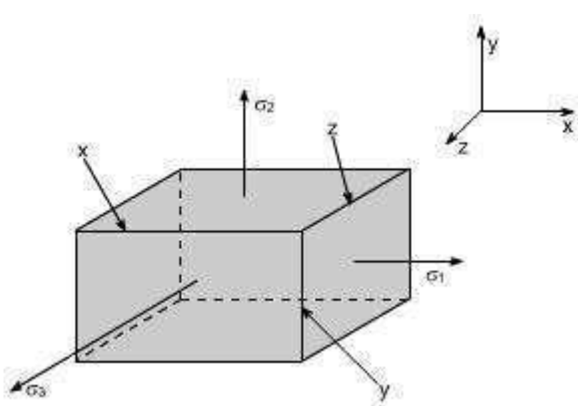
$$\epsilon_1 .E = \sigma_1 - \mu\sigma_2$$
$$\epsilon_2 .E = \sigma_2 - \mu\sigma_1$$

so the solution of above two equations yields

$$\boxed{\begin{aligned}\sigma_1 &= \frac{E}{(1-\mu^2)}[\epsilon_1 + \mu\epsilon_2] \\ \sigma_2 &= \frac{E}{(1-\mu^2)}[\epsilon_2 + \mu\epsilon_1]\end{aligned}}$$

Hydrostatic stress : The term Hydrostatic stress is used to describe a state of tensile or compressive stress equal in all directions within or external to a body. Hydrostatic stress causes a change in volume of a material, which if expressed per unit of original volume gives a volumetric strain denoted by Îv. So let us determine the expression for the volumetric strain.

Volumetric Strain:



Consider a rectangle solid of sides x, y and z under the action of principal stresses s_1 , s_2 , s_3 respectively.

Then $Î_1$, $Î_2$, and $Î_3$ are the corresponding linear strains, than the dimensions of the rectangle becomes

(x + $Î_1$. x); (y + $Î_2$. y); (z + $Î_3$. z)

hence the

$$\text{Volumetric strain} = \frac{\text{Increase in volume}}{\text{Original volume}}$$

$$= \frac{x(1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3)z - xyz}{xyz}$$

$$= (1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3) - 1 \cong \epsilon_1 + \epsilon_2 + \epsilon_3 \left[\text{Neglecting the products of } \epsilon\text{'s}\right]$$

ALITER: Let a cuboid of material having initial sides of Length x, y and z. If under some load system, the sides changes in length by dx, dy, and dz then the new volume (x + dx) (y + dy) (z +dz)

New volume = xyz + yzdx + xzdy + xydz

Original volume = xyz

Change in volume = yzdx +xzdy + xydz

Volumetric strain = (yzdx +xzdy + xydz) / xyz = $Î_x$+ $Î_y$+ $Î_z$

Neglecting the products of epsilon's since the strains are sufficiently small.

Volumetric strains in terms of principal stresses:

As we know that

$$\epsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

$$\text{Futher Volumetric strain} = \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{E} - \frac{2\mu(\sigma_1 + \sigma_2 + \sigma_3)}{E}$$

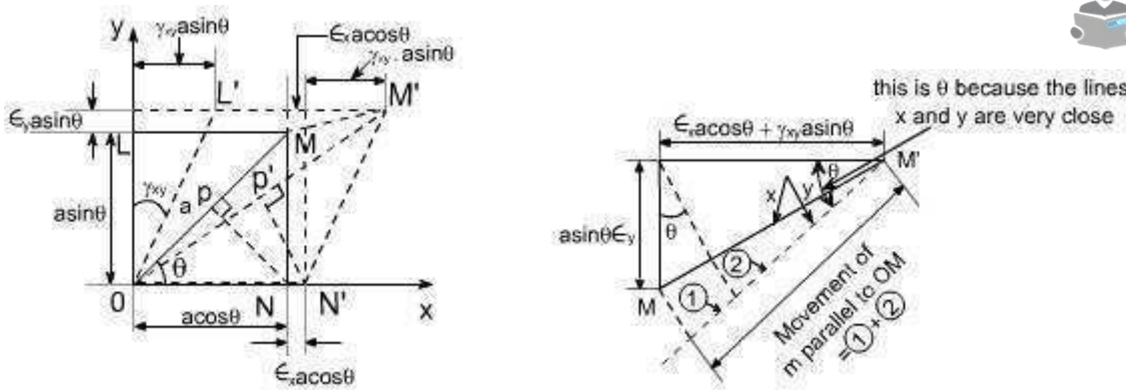
$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}$$

hence the

$$\boxed{\text{Volumetric strain} = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}}$$

Strains on an oblique plane

(a) Linear strain



Consider a rectangular block of material OLMN as shown in the xy plane. The strains along ox and oy are Î$_x$ and Î$_y$, and g$_{xy}$ is the shearing strain.

Then it is required to find an expression for Î$_q$, i.e the linear strain in a direction inclined at q to OX, in terms of Î$_x$,Î$_y$, g$_{xy}$ and q.

Let the diagonal OM be of length 'a' then ON = a cos q and OL = a sin q , and the increase in length of those under strains are Î$_x$acos q and Î$_y$a sin q (i.e. strain x original length) respectively.

If M moves to M', then the movement of M parallel to x axis is Î$_x$acos q + g$_{xy}$ sin q and the movement parallel to the y axis is Î$_y$asin q

Thus the movement of M parallel to OM, which since the strains are small is practically coincident with MM'. and this would be the summation of portions (1) and (2) respectively and is equal to

$$= (\epsilon_y \ a\sin\theta)\sin\theta + (\epsilon_x \ a\cos\theta + \gamma_{xy} a\sin\theta)\cos\theta$$

$$= a\left[\epsilon_y \sin\theta.\sin\theta + \epsilon_x \cos\theta.\cos\theta + \gamma_{xy}\sin\theta.\cos\theta\right]$$

hence the strain along OM

$$= \frac{\text{extension}}{\text{original length}}$$

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \gamma_{xy}\sin\theta.\cos\theta + \epsilon_y \sin^2\theta$$

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy}\sin\theta.\cos\theta$$

$$\text{Recalling } \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\text{or } 2\cos^2\theta - 1 = \cos 2\theta$$

$$\cos^2\theta = \left[\frac{1+\cos 2\theta}{2}\right]$$

$$\sin^2\theta = \left[\frac{1-\sin 2\theta}{2}\right]$$

hence

$$\epsilon_\theta = \epsilon_x\left[\frac{1+\cos 2\theta}{2}\right] + \epsilon_y\left[\frac{1-\sin 2\theta}{2}\right] + \gamma_{xy} a\sin\theta.\cos\theta$$

$$= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\epsilon_\theta = \left\{\frac{\epsilon_x + \epsilon_y}{2}\right\} + \left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

This expression is identical in form with the equation defining the direct stress on any inclined plane q with Î$_x$ and Î$_y$ replacing s$_x$ and s$_y$ and ½ g$_{xy}$ replacing t$_{xy}$ i.e. the shear stress is replaced by half the shear strain

Shear strain: To determine the shear stain in the direction OM consider the displacement of point P at the foot of the perpendicular from N to OM and the following expression can be derived as

$$\frac{1}{2}\gamma_\theta = -\left[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta\right]$$

In the above expression ½ is there so as to keep the consistency with the stress relations.

Further -ve sign in the expression occurs so as to keep the consistency of sign convention, because OM' moves clockwise with respect to OM it is considered to be negative strain.

The other relevant expressions are the following:

Principal planes :

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

Principal strains :

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

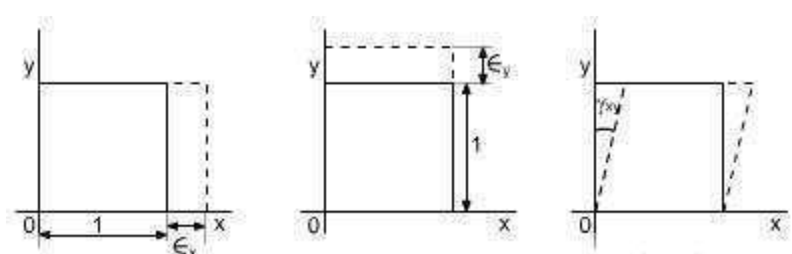
Maximum shear strains :

$$\frac{\gamma_{max}}{2} = \pm\sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Let us now define the plane strain condition

Plane Strain:

In xy plane three strain components may exist as can be seen from the following figures:



(Fig.1) (Fig.2) (Fig.3)

Therefore, a strain at any point in body can be characterized by two axial strains i.e Î$_x$ in x direction, Î$_y$ in y - direction and g$_{xy}$ the shear strain.

In the case of normal strains subscripts have been used to indicate the direction of the strain, and Î$_x$, Î$_y$ are defined as the relative changes in length in the co-ordinate directions.

With shear strains, the single subscript notation is not practical, because such strains involve displacements and length which are not in same direction. The symbol and subscript g$_{xy}$ used for the shear strain referred to the x and y planes. The order of the subscript is unimportant. g$_{xy}$ and g$_{yx}$ refer to the same physical quantity. However, the sign convention is important. The shear strain g$_{xy}$ is considered to be positive if it represents a decrease the angle between the sides of an element of material lying parallel the positive x and y axes. Alternatively we can think of positive shear strains produced by the positive shear stresses and vice versa.

Plane strain:

An element of material subjected only to the strains as shown in Fig. 1, 2, and 3 respectively is termed as the plane strain state.

Thus, the plane strain condition is defined only by the components Î$_{x}$, Î$_{y}$, g$_{xy}$: Î$_{z}$ = 0; g$_{xz}$= 0; g$_{yz}$= 0

It should be noted that the plane stress is not the stress system associated with plane strain. The plane strain condition is associated with three dimensional stress system and plane stress is associated with three dimensional strain system.

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